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Pseudodifferential Parabolic Equations of Sound Propagation in the Slowly Range-Dependent Ocean and Their Numerical Solutions

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Abstract – This paper offers a mathematical model of low-frequency harmonic sound propagation in the ocean with $2D$ and $3D$ inhomogeneities. The sound speed, density and bottom relief slowly vary with horizontal coordinates and arbitrarily with depth, including an inhomogeneous bottom. A one-way (parabolic) approximation is derived, and a numerical algorithm for its implementation is offered, ensuring any given accuracy of calculations for any given interval of grazing angles of local normal modes (superwide-angle property) for any parameters of the medium.

As a first step, let us consider the technique of additive factoring for the acoustics equations in a stationary liquid medium with two-dimensional inhomogeneities.

Harmonic sound propagation in a three-dimensional liquid half-space translationally invariable on the Cartesian y coordinate is described by the following system of differential equations (the first two are linearized Newton equations and the third is a linearized continuity equation) [1],

$$\begin{vmatrix} D_x & 0 & -i\omega\rho \\ D_z & -i\omega\rho & 0 \\ -i\omega\rho^{-1}C^{-2} & D_z & D_x \end{vmatrix} \begin{vmatrix} p \\ v_z \\ v_x \end{vmatrix} = \begin{vmatrix} f_x \\ f_z \\ V \end{vmatrix}, \qquad (1)$$

$$(x, z) \in (-\infty, \infty) \times (0, \infty)$$

and by the condition of limiting absorption

$$\operatorname{Im} C < 0 \Rightarrow |p(x, z)| < M, \quad (x, z) \in (-\infty, \infty) \times (0, \infty).$$

Here, p is the acoustical pressure, v_x and v_z are the Cartesian acoustical velocity components, f_x and f_z are the Cartesian components of the external force density, V is the external volume velocity density, ρ is the medium's density, C is medium's sound speed (all these quantities are functions of both cartesian coordinates x and y), and ω is the cyclic frequency. D_x and D_z are the partial derivatives with respect to space coordinates, i is the imaginary unity, and M is a number. We transform (1) by premultiplying these equations by the nonsingular matrix

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & D_z(i\omega\rho)^{-1} & 1 \end{vmatrix}.$$

This yields a system of equations involving only acoustical pressure and the horizontal component of acoustical velocity

$$\begin{vmatrix} D_x\mathbf{p} \\ D_x\mathbf{v} \end{vmatrix} = \begin{vmatrix} 0 & i\omega\mathbb{R} \\ i\omega\mathbb{B} & 0 \end{vmatrix} \begin{vmatrix} \mathbf{p} \\ \mathbf{v} \end{vmatrix} + \begin{vmatrix} \mathbf{f}_x \\ \mathbf{w} \end{vmatrix},$$

$$\begin{gathered} \mathbf{p}(x) = p(x, z), \quad \mathbf{v}_x(x) = v_x(x, z), \\ \mathbb{R}\mathbf{v} = \rho(x, z)v_x(x, z), \\ \mathbb{B}\mathbf{p}(x) = (\rho^{-1}(x, z)C^{-2}(x, z) \\ - D_z((i\omega)^2\rho(x, z))^{-1}D_z)p(x, z), \\ \mathbf{f}_z(x) = f_z(x, z), \ \mathbf{w}(x) = V(x, z) - D_z(i\omega\rho(x, z))^{-1}f_z(x, z). \end{gathered} \qquad (2)$$

Operator equations (2) are the basis-invariant form of guided-wave equations [2]: assuming for the sake of simplicity $\mathbb{R} = \mathbb{I}$ and taking into account that on the basis of local normal modes $\phi_l(x, z)$ the operator $\mathbb{B}$ has the diagonal form

$$\mathbb{B}(x) = \hat{\Phi}(x)\mathbb{C}^{-2}(x)\hat{\Phi}^{-1}(x), \quad \hat{\Phi}(x) = \underset{l}{\text{row}}\,\{\vec{\phi}_l(x)\},$$

$$\mathbb{C}(x) = \underset{l}{\text{diag}}\,\{C_l\},$$

where C_l denotes the phase velocity of the lth local normal mode, we obtain for the amplitudes of the local normal modes of pressure $\mathbf{a}(x) = \hat{\Phi}^{-1}(x)\mathbf{p}(x)$ and velocity $\mathbf{b}(x) = \hat{\Phi}^{-1}(x)\mathbf{v}(x)$ the equations:

$$D_x a_l(x) - i\omega b_l(x) = -\sum_k \gamma_{lk}(x)a_k(x),$$

$$D_x b_l(x) - i\omega C_l^{-2}(x)a_l(x) = -\sum_k \gamma_{lk}(x)b_l(x),$$

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where $\gamma_{lk} = [\hat{\Phi}^{-1}(x)(D_x\hat{\Phi}(x))]_{lk}$ are the coefficients of coupling of the local normal modes.

Let us now reformulate (2) in the manner of the Wentzel–Kramer–Brillouin method or the additive factoring of the solution to the form of waves propagating to the left and right. Let us assume that outside the interval of the horizontal coordinate $x \in (a, b)$ our waveguide is layered and all sources are inside this interval. Introducing the new local amplitudes of waves propagating to the left and right:

$$\begin{vmatrix} \mathbf{p} \\ \mathbf{v} \end{vmatrix} = \begin{vmatrix} \mathbb{Y}^{-1}(x) & \mathbb{Y}^{-1}(x) \\ \mathbb{Z}(x) & -\mathbb{Z}(x) \end{vmatrix} \begin{vmatrix} \mathbf{u}^+ \\ \mathbf{u}^- \end{vmatrix}, \quad (3)$$

$$\mathbb{S}(x) = (\sqrt{\mathbb{R}(x)}\,\mathbb{B}(x)\sqrt{\mathbb{R}(x)})^{1/2},$$

$$\mathbb{Y}(x) = \sqrt{\mathbb{S}(x)}\sqrt{\mathbb{R}(x)}^{-1},\ \mathbb{Z}(x) = \sqrt{\mathbb{R}(x)}^{-1}\sqrt{\mathbb{S}(x)},$$

and taking into account that only outgoing waves may exist outside of the interval (a, b), we obtain by the process of additive factoring [3, 4] the following system of equations:

$$\begin{vmatrix} \mathbb{P} & \mathbb{U} \\ \mathbb{L} & \mathbb{Q} \end{vmatrix} \begin{vmatrix} \mathbf{u}^+ \\ \mathbf{u}^- \end{vmatrix} = \begin{vmatrix} \mathbf{F}^+ \\ \mathbf{F}^- \end{vmatrix},$$

$$\mathbb{P} = D_x - i\omega\mathbb{S}(x)$$
$$+ 1/2\{\mathbb{Z}^{-1}(x)(D_x\mathbb{Z}(x)) - \mathbb{Y}^{-1}(x)[D_x\mathbb{Y}(x)]\},$$
$$\mathbf{u}^+(a) = 0, \quad (4)$$
$$\mathbb{Q} = D_x + i\omega\mathbb{S}(x)$$
$$+ 1/2\{\mathbb{Z}^{-1}(x)(D_x\mathbb{Z}(x)) - \mathbb{Y}^{-1}(x)[D_x\mathbb{Y}(x)]\},$$
$$\mathbf{u}^-(b) = 0,$$
$$\mathbb{L} = \mathbb{U} = -1/2\{\mathbb{Z}^{-1}(x)(D_x\mathbb{Z}(x)) + \mathbb{Y}^{-1}(x)[D_x\mathbb{Y}(x)]\},$$

$$\begin{vmatrix} \mathbf{F}^+ \\ \mathbf{F}^- \end{vmatrix} = 1/2 \begin{vmatrix} \mathbb{Y} & \mathbb{Z}^{-1} \\ \mathbb{Y} & -\mathbb{Z}^{-1} \end{vmatrix} \begin{vmatrix} \mathbf{f}_x \\ \mathbf{w} \end{vmatrix}.$$

It can be solved by the Gauss–Zeidel iterations [5, 6]:

$$\mathbf{u}_k^+ = \mathbb{P}^{-1}(-\mathbb{U}\mathbf{u}_{k-1}^- + \mathbf{F}^+),\quad \mathbf{u}_k^- = \mathbb{Q}^{-1}(-\mathbb{L}\mathbf{u}_k^+ + \mathbf{F}^-) \quad (5)$$

converging, if $\|\mathbb{P}^{-1}\mathbb{U}\mathbb{Q}^{-1}\mathbb{L}\| < 1$, where $\|\cdot\|$ stands for the norm. For ρ and C slowly varying with x, the operators $\mathbb{U}$ and $\mathbb{L}$ are small; and, for well-defined Cauchy operators $\mathbb{P}$ and $\mathbb{Q}$, the likelihood of convergence of the process (5) is very high. From (5), we can see that the operators $\mathbb{L}$ and $\mathbb{U}$ describe the backscattering process, whereas the operators $\mathbb{P}$ and $\mathbb{Q}$ describe propagation (terms $\pm i\omega\mathbb{S}$) and transmission through the horizontal inhomogeneities.

Assuming now the zero field as the zero order approximation, we obtain for the first order approximation of one-way waves in positive and negative directions $\mathbf{u}_1^+$ and $\mathbf{u}_1^-$ the equations:

$$\mathbb{P}\mathbf{u}_1^+ = \mathbf{F}^+,\quad \mathbb{Q}\mathbf{u}_1^- = \mathbf{F}^-.$$

In the basis of local normal modes, they assume the known form of one-way guided waves (see [2])

$$D_x c_l^+ - i\omega C_l^{-1} c_l^+$$
$$= -\sum_k \gamma_{lk}(C_l + C_k)\,(2\sqrt{C_lC_k})^{-1} + \{\Phi^{-1}\mathbf{F}^+\}_l,$$

$$D_x c_l^- + i\omega C_l^{-1} c_l^-$$
$$= -\sum_k \gamma_{lk}(C_l + C_k)\,(2\sqrt{C_lC_k})^{-1} + \{\Phi^{-1}\mathbf{F}^+\}_l.$$

Assuming the coupling coefficients γ_{lk} to be rapidly diminishing with the difference of numbers of local normal modes $|l - k|$, and, therefore, letting (see [2])

$$(C_l + C_k)/(2\sqrt{C_lC_k})^{-1} \cong 1$$

thus neglecting the assumption that the transmission through horizontal inhomogeneities is equal to unity, we obtain in operator form the abstract pseudodifferential parabolic equations (PEs)

$$D_x\mathbf{u}_1^+ - i\omega\mathbb{S}(x)\mathbf{u}_1^+ = \mathbf{F}^+,\ D_x\mathbf{u}_1^- + i\omega\mathbb{S}(x)\mathbf{u}_1^- = \mathbf{F}^-. \quad (6)$$

These equations form a basis to obtain all known PEs by the technique employed to approximate $\mathbb{S}$, and to solve the resulting approximated differential equation.

It should be noted that if the operator $\mathbb{B}$ is selfadjoint, the factoring process (3), (4) conserves the flow form of energy

$$\mathbf{p}^*\mathbf{v} = (\mathbb{Y}^{-1}(\mathbf{u}^+ + \mathbf{u}^-))^*(\mathbb{Z}(\mathbf{u}^+ - \mathbf{u}^-))$$
$$= (\mathbf{u}^+)^*\mathbf{u}^+ - (\mathbf{u}^-)^*\mathbf{u}^-.$$

Moreover, this factoring provides the exact reciprocity of the first order equations for right- and left-going waves

$$\mathbb{P}\mathbf{u}_1^+ = \mathbf{F}^+$$

and

$$\mathbb{Q}\mathbf{u}_1^- = \mathbf{F}^-.$$

However, in practical calculations, we may sacrifice some accuracy in order to diminish computational time. If sound pressure alone is needed, then the simplicity of the relationship

$$\mathbf{p} = \mathbf{u}^+ + \mathbf{u}^-$$

make interesting the factoring

$$\begin{vmatrix} \mathbf{p} \\ \mathbf{v} \end{vmatrix} = \begin{vmatrix} \mathbb{I} & \mathbb{I} \\ \mathbb{Z}(x) & -\mathbb{Z}(x) \end{vmatrix} \begin{vmatrix} \mathbf{u}^+ \\ \mathbf{u}^- \end{vmatrix},$$

$$\mathbb{S}(x) = \sqrt{\mathbb{R}(x)\,\mathbb{B}(x)},$$

$$\mathbb{Z}(x) = \mathbb{R}^{-1}(x)\mathbb{S}(x) = \mathbb{B}(x)\mathbb{S}^{-1}(x).$$

Equations (4) take then the form

$$\begin{vmatrix} \mathbb{P} & \mathbb{U} \\ \mathbb{L} & \mathbb{Q} \end{vmatrix} \begin{vmatrix} \mathbf{u}^+ \\ \mathbf{u}^- \end{vmatrix} = \begin{vmatrix} \mathbf{F}^+ \\ \mathbf{F}^- \end{vmatrix},$$

$$\mathbb{P} = D_x - i\omega\mathbb{S}(x) + \frac{1}{2}\mathbb{Z}^{-1}(x)[D_x\mathbb{Z}(x)],$$

$$\mathbf{u}^+(a) = 0,$$

$$\mathbb{Q} = D_x + i\omega\mathbb{S}(x) + \frac{1}{2}\mathbb{Z}^{-1}(x)[D_x\mathbb{Z}(x)],$$

$$\mathbf{u}^-(b) = 0,$$

$$\mathbb{L} = \mathbb{U} = -\frac{1}{2}\mathbb{Z}^{-1}(x)[D_x\mathbb{Z}(x)],$$

$$\begin{vmatrix} \mathbf{F}^+ \\ \mathbf{F}^- \end{vmatrix} = \begin{vmatrix} \mathbb{I} & \mathbb{Z}^{-1} \\ \mathbb{I} & -\mathbb{Z}^{-1} \end{vmatrix} \begin{vmatrix} \mathbf{f}_x \\ \mathbf{w} \end{vmatrix},$$

which again yields (6), if we neglect the difference of the transmission coefficient from unity. The solutions of these equations satisfy the reciprocity principle.

In the case of a three-dimensional cylindrically symmetrical ocean with the source located on the symmetry axis, equations (1) take the form

$$\begin{vmatrix} D_r & 0 & -i\omega\rho \\ D_z & -i\omega\rho & 0 \\ -i\omega\rho^{-1}C^{-2} & D_z & r^{-1}D_r r \end{vmatrix} \begin{vmatrix} p \\ v_z \\ v_r \end{vmatrix} = \begin{vmatrix} 0 \\ f_z 2r^{-1}\delta(r) \\ V2r^{-1}\delta(r) \end{vmatrix},$$

$$(r, z) \in (0, \infty) \times (0, H)$$

and may be treated as the previous problem by substituting $r = \exp(x)$, $\mathbf{w} = \exp(x)v_r$, and eliminating v_z for $r > 0$

$$\begin{vmatrix} D_x\mathbf{p} \\ D_x\mathbf{v} \end{vmatrix} = \begin{vmatrix} 0 & i\omega\mathbb{R} \\ i\omega\exp(2x)\mathbb{B} & 0 \end{vmatrix} \begin{vmatrix} \mathbf{p} \\ \mathbf{w} \end{vmatrix}.$$

After returning to r the one-way equation similar to (6) will have the form

$$D_r\mathbf{p}_1^+ - i\omega\,\mathbb{S}(r)\mathbf{p}_1^+ + r^{-1}/2 = 0$$

or

$$\mathbf{p}_1^+ = r^{-1/2}\mathbf{q},$$

$$D_r\mathbf{q}^+ - i\omega\,\mathbb{S}(r)\mathbf{q}^+ = 0. \tag{7}$$

Assuming the waveguide to be layered in the vicinity of the source up to the distances where the simplest asymptotic expansion of Hankel function

$$H_0^{(1)}(\omega s r) \approx (2/i\pi\omega s r)^{1/2}\exp(i\omega s r)$$

becomes valid, and taking into account that it satisfies (7), we obtain the initial condition for $\mathbf{q}$ as

$$\mathbf{q}(0) = (\omega\mathbb{S}/2\pi i)^{-1/2}\mathbf{w}(z). \tag{8}$$

Equations (7) and (8) fully determine the field of the wave outgoing from the source.

To solve (6) numerically, we use the exponential fitting algorithm [7, 8]:

$$\mathbf{u}^+(x + h) = \exp[i\omega h\mathbb{S}(x)]\mathbf{u}^+(x).$$

The action of the propagation factor operator

$$\mathbb{E} = \exp[i\omega h\mathbb{S}(x)]$$

can be calculated with any given accuracy with use of the rational-fraction approximations [9] if

$$f(\lambda) \cong F_n(\lambda)/G_m(\lambda)$$

(F_n, G_m are the polynomials of degree n, m) in the vicinity of the spectrum of the operator $\mathbb{S}$, then the Riesz definition of the function f of the operator $\mathbb{T}$

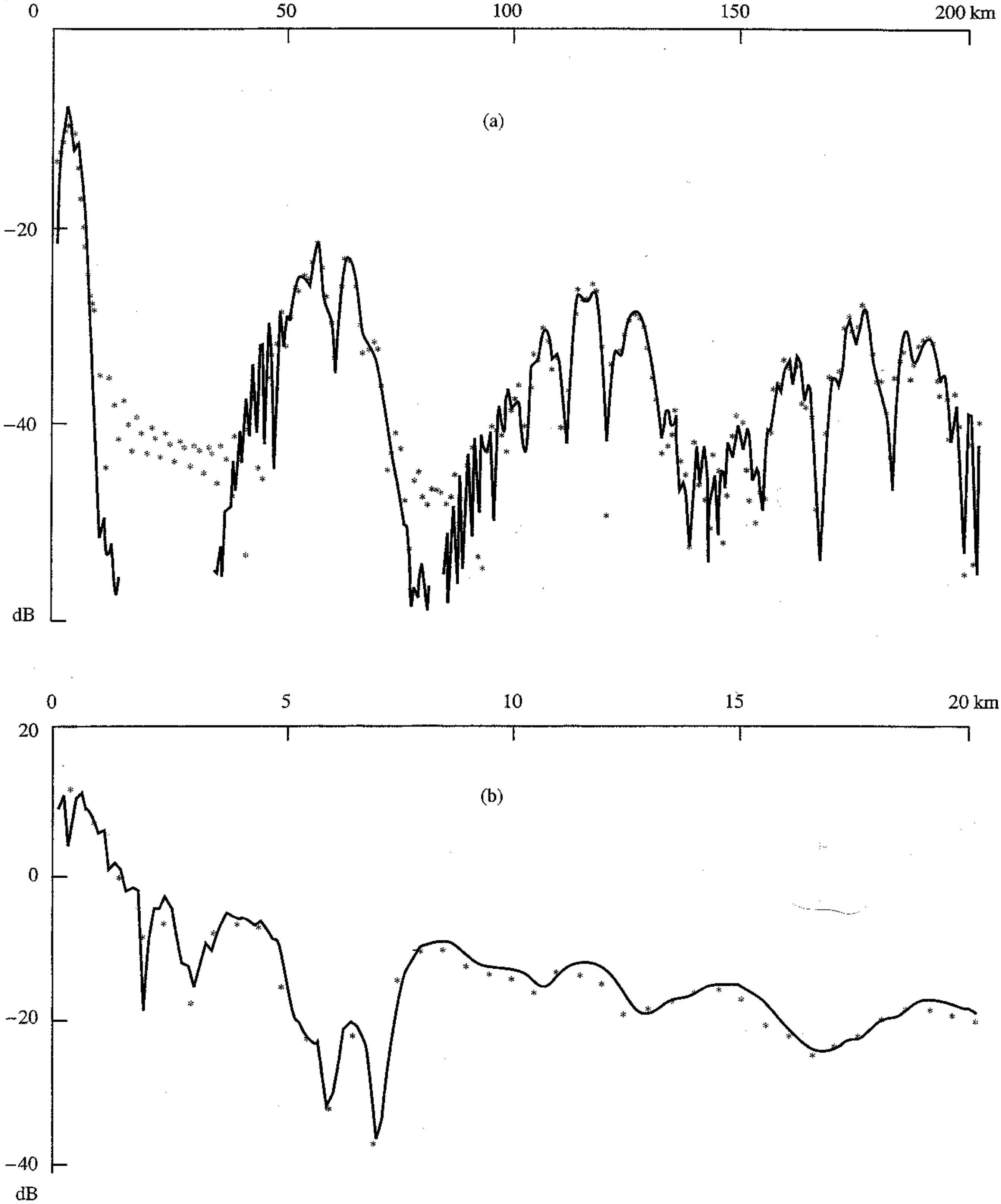
$$f(\mathbb{T}) = (2\pi i)^{-1}\int_\Gamma f(\lambda)(\mathbb{T} - \lambda\mathbb{I})^{-1}d\lambda,$$

where $\mathbb{I}$ is the identity operator, the integration path Γ encircles the spectrum of the operator $\mathbb{T}$ leaving him on the left, and $\mathbb{T}$ is assumed such that the integral does converge (see [10]), leading us to the conclusion that

$$f(\mathbb{S}) \cong F_n(\mathbb{S})(G_m(\mathbb{S}))^{-1}.$$

We construct the appropriate approximation to $\exp(ih\sqrt{\lambda})$ with the known Padé approximations [11] of the exponent

$$\exp(\lambda) = (M_n(\lambda^2) + \lambda N_n(\lambda^2)) \times (M_n(\lambda^2) - \lambda N_n(\lambda^2))^{-1} + E_n(\lambda)$$



Comparison of sound field calculations by the pseudodifferential parabolic equation technique (solid line) and by the normal mode method (*). (a) Deep ocean: Atlantic profile, ocean depth 5.5879 km, frequency 25 Hz, duct axis depth 1.015 km, source depth 0.2539 km, and receiver depth 0.8655 km. (b) Shallow sea: depth 0.24 km, frequency 100 Hz, duct axis depth 0.12 km, source depth 0.03 km, and receiver depth 0.09 km.

and of the square root

$$\lambda^{1/2} = F_m(\lambda)(G_m(\lambda)^{-1} + H_m(\lambda),$$

where M_n, N_n, F_m, and G_m are the polynomials with real coefficients, computed by recurrence, and E_m and H_n are the approximation errors known to fade with increasing m and n in the complex plane with the cut on the negative side of the real axes. Combining those approximations, we get

$$\begin{aligned}&\exp(ih\lambda^{1/2})\\ &= (G_m(\lambda)M_n(-h^2\lambda) + ihF_m(\lambda)N_n(-h^2\lambda))\\ &\times (G_m(\lambda)M_n(-h^2\lambda) - ihF_m(\lambda)N_n(-h^2\lambda))^{-1} \quad (9)\\ &= \prod_{k=1}^{k=n/2+m} (\lambda - \mu_k^*(h))\,(\lambda - \mu_k(h))^{-1},\end{aligned}$$

where μ_k are the roots of the denominator of the above fraction and μ_k^* are their complex conjugates. We have computed μ_k for some m, n, and h. All of them turned out to lie in the fourth quadrant of the complex plane far enough from the spectrum of $\mathbb{S}^2$ lying in the first and second quadrants, which implies the stability of the action of the operator $\mathbb{E}$. The error of the approximation in (9) decreases with increasing n and m in the whole complex plane without the negative half of the real axes, but it is important to approximate the region of spectrum of $\mathbb{S}^2$, corresponding to propagating local normal modes, lying in the vicinity of the point (1, 0) in the upper half-plane. For example, in the local normal modes with Brillouin angles up to 60°, $m = 4$, $n = 4$, and $h = 4\pi$ give a phase error less than 5×10^{-6} rad, thus enabling the calculation of underwater sound propagation to a distance of 9000 km with frequencies up to 100 Hz with an absolute phase error less then $\pi/2$.

Let us transform the approximation of (9) of the operator $\mathbb{E}$

$$\begin{aligned}&\prod_{k=1}^{k=n/2+m} (\mathbb{R}\mathbb{B} - \mu_k^*(h))\,(\mathbb{R}\mathbb{B} - \mu_k(h))^{-1} \quad (10)\\ &= \prod_{k=1}^{K=n/2+m} (\mathbb{I} - 2\mathrm{Im}(\mu_k)\,(\mathbb{B} - \mu_k\mathbb{R}^{-1})^{-1}\mathbb{R}^{-1}).\end{aligned}$$

It can be seen from (10) that, to act by the operator $\mathbb{E}$, it is sufficient to solve K boundary problems of the type

$$\begin{aligned}&(\rho^{-1}(x,z)C^{-2}(x,z) - \mu_k\rho^{-1}(x,z)\\ &- D_z((i\omega)^2\rho(x,z))^{-1}D_z)\,q_k(z)\\ &= \rho^{-1}(x,z)q_{k-1}(z),\\ &q_0(z) = p(x,z),\ \ q_K(z) = p(x+h,z)\end{aligned}$$

with some boundary conditions. If the square of the horizontal wavenumber is present in the boundary conditions as for the liquid half-space or a solid layer on the ocean surface, then it must be replaced with μ_k. These boundary problems can be solved on digital computers by using various discrete techniques. The finite elements technique in the form of integral equalities [12] seems to combine the simplicity of implementation and a high enough order of approximation of the original differential problem due to the special form

$$\mathbb{B} - \mu_k\mathbb{R}^{-1} \approx \hat{M}^{-1}\hat{L}.$$

Here, $\hat{M}$ and $\hat{L}$ are the band matrices easily constructed from the coefficients of the original differential problem.

Based on the above algorithms, we have developed a package of Fortran-coded routines to calculate long-range underwater sound propagation in the range-depth dependent environments. The performance of this package on the IBM PC may be characterized as follows. On a computer with a speed of 3.425×10^6 DWHET (double precision Whetstone test from MsFortran package), the calculation for the 10-km thick of ocean-bottom model at the frequency 25 Hz up to a range of 200 km takes 580 s with sufficient accuracy in the grazing angle interval up to ±35°. The calculation results for layered waveguides are practically identical to the results obtained by the normal modes technique. The figure shows the results for two benchmarks from [13] for a deep and a shallow sea. Comparison of different calculation techniques and applications of our technique to the interpretation of experimental data can be found in [14, 15].

Let us now consider an implementation of the full iteration process (5) evaluating the wavefields in both directions. To avoid the calculation of the derivatives of medium properties, from the very beginning, we use a medium model consisting of short, layered, waveguide intervals ajoining each other at vertical sections. Using the lower index l to denote the waveguide interval (x_{l-1}, x_l), and denoting

$$\begin{aligned}&E_l = \exp(i\omega S_l(x_l - x_{l-1})),\\ &S_l = (\mathbb{R}(x_l)\,\mathbb{B}(x_l))^{1/2},\quad Z_l = \mathbb{R}^{-1}(x_l)S_l,\\ &x_0 = a,\ \ x_L = b,\end{aligned}$$

we obtain from the conditions of continuity on the borders of the intervals (omitting the intermediate calculations) the operators $\mathbb{P}$, $\mathbb{Q}$ as two-diagonal matrices, and $\mathbb{L}$, $\mathbb{U}$ as diagonal matrices:

$$
\mathbb{P} = \begin{vmatrix} 1 & 0 & & & & \\ -1 & (1+Z_0^{-1}Z_1)/2 & & & & \\ 0 & -E_1 & (1+Z_0^{-1}Z_2)/2 & & & \\ & & \dots & & & \\ & & -E_{L-1} & (1+Z_{L-1}^{-1}Z_L)/2 & 0 & \\ & & & -E_L & (1+Z_L^{-1}Z_{L+1})/2 & \end{vmatrix},
$$

$$
\mathbb{Q} = \begin{vmatrix} -(1+Z_1^{-1}Z_0)/2 & E_1 & & & \\ 0 & -(1+Z_2^{-1}Z_1)/2 & E_2 & & \\ & & \dots & & \\ & & & -(1+Z_{L+1}^{-1}Z_L)/2 & 1 \\ & & & 0 & 1 \end{vmatrix}, \tag{11}
$$

$$
\mathbb{L} = \begin{vmatrix} (Z_1^{-1}Z_0-1)/2 & 0 & & & \\ 0 & (Z_2^{-1}Z_1-1)/2 & & & \\ & & \dots & & \\ & & & (Z_{L+1}^{-1}Z_L-1)/2 & 0 \\ & & & 0 & 0 \end{vmatrix},
$$

$$
\mathbb{U} = \begin{vmatrix} 0 & 0 & & & \\ 0 & -(Z_0^{-1}Z_1-1)/2 & & & \\ & & \dots & & \\ & & & -(Z_{L-1}^{-1}Z_L-1)/2 & 0 \\ & & & 0 & -(Z_L^{-1}Z_{L+1}-1)/2 \end{vmatrix},
$$

whose elements do not use the derivatives of medium properties. The action of Z and Z^{-1} can be fulfilled using the Padé approximations. The one-way algorithm, including the transmission operator, takes the form

$$
\mathbf{u}_{l+1} = 2(1+Z_l^{-1}Z_{l+1})^{-1}E_l\mathbf{u}_l.
$$

For small differences between Z_l and Z_{l+1}, in a first approximation, we obtain from this difference an easy recurrence equation

$$
\mathbf{u}_{l+1} = (3/2-1/2Z_l^{-1}Z_{l+1})E_l\mathbf{u}_l.
$$

Let us now construct the algorithm for the calculation of the one-way approximation sound field in the ocean, with three-dimensional inhomogeneities slowly varying along the horizontal coordinates. The system of acoustic equations for a three-dimensional stationary liquid has the form

$$
\begin{vmatrix} D_x & 0 & 0 & -i\omega\rho \\ D_y & 0 & -i\omega\rho & 0 \\ D_z & -i\omega\rho & 0 & 0 \\ -i\omega\rho^{-1}C^{-2} & D_z & D_y & D_x \end{vmatrix} \begin{vmatrix} p \\ v_z \\ v_y \\ v_x \end{vmatrix} = \begin{vmatrix} f_x \\ f_z \\ f_y \\ V \end{vmatrix},
$$

$$
Y_0p(x, y, 0) + v_z(x, y, 0) = 0,
$$

$$
Y_Hp(x, y, H) + v_z(x, H) = 0,
$$

$$
(x, y, z) \in (-\infty, \infty) \times (-\infty, \infty) \times (0, H).
$$

Eliminating the vertical and shear components of the acoustical velocities v_z and v_y, we obtain their operator form:

$$\begin{vmatrix} D_x\mathbf{p} \\ D_x\mathbf{v} \end{vmatrix} = \begin{vmatrix} 0 & i\omega\mathbb{R} \\ i\omega\mathbb{B} & 0 \end{vmatrix} \begin{vmatrix} \mathbf{p} \\ \mathbf{v} \end{vmatrix} + \begin{vmatrix} \mathbf{f}_x \\ \mathbf{w} \end{vmatrix}.$$

Assuming that the medium properties slowly vary along the x axis, with the additive factoring technique we get the one-way equation (neglecting the difference of the transmission coefficient from unity)

$$D_x\mathbf{p}_1^+ - i\omega\,\mathbb{S}(x)\mathbf{p}_1^+ = 0,$$

$$\mathbb{S} = (C^{-2} - \rho D_y((i\omega)^2\rho)^{-1}D_y - \rho D_z((i\omega)^2\rho)^{-1}D_z)^{1/2}.$$

Using the algorithm (10) to solve this equation yields

$$E \cong \prod_k (I - 2\mathrm{Im}(\mu_k) \times (C^{-2} - \rho D_y((i\omega)^2\rho)^{-1}D_y - \rho D_z((i\omega)^2\rho)^{-1}D_z - \mu_k)^{-1}). \tag{12}$$

From (12), we see that, to calculate the field of acoustical pressure on each step on the x axis, we need to solve two-dimensional boundary problems analogous to the two-dimensional acoustic boundary problems in a highly absorbing medium because all $\mathrm{Im}\,\mu_k < 0$:

$$\begin{vmatrix} D_y & 0 & -i\omega\rho \\ D_z & -i\omega\rho & 0 \\ -i\omega\rho^{-1}(C^{-2} - \mu_k) & D_z & D_y \end{vmatrix} \begin{vmatrix} p \\ v_z \\ v_y \end{vmatrix} = \begin{vmatrix} f_x \\ f_z \\ V \end{vmatrix},$$

$$Y_0 p(y, 0) + v_z(y, 0) = 0, \quad Y_H p(y, H) + v_z(y, H) = 0.$$

If the properties of the medium vary slowly also along the y axis, then this equation can be solved by applying the iteration process in the form (11) describing the propagation of waves in both directions.

Thus, the use of the additive factoring technique and the rational-fractions Padé approximations in the calculation of the harmonic sound field in two- and three-dimensional models of the ocean yielded algorithms and, partly, programs offering sufficient performance and accuracy.

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