Additive factorization + Pade approximations = effective underwater sound calculations

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March 3, 1994

The family of algorithms for harmonical and banded sound field calculations in $2D$ and $3D$ ocean models with sound speed and density slowly varying with horizontal coordinates and arbitrary on depth including nonhomogenuous solid bottom is proposed, based on the method of additive WKB-like factoring. The accordance between such a method and the coupled modes two- and one-way solutions is established in $2D$ case. Convergence of the appropriate iteration process is evaluated. The resulting initial pseudodifferential parabolic problems are numerically solved with any given accuracy by the high order Pade-type approximation algorithms, featuring several wavelengths step size on horizontal coordinates. The performance of the computer implementation, adopted to underwater sound propagation calculations in $2D$ model is discussed.

PACS numbers: 43.20Bi, $43.20Mv, 43.30Bp$

INTRODUCTION

Many problems of underwater sound propagation and marine seismology are of interest in case not only of layered media but of media with properties varying in two or three directions. Among mathematical procedures to predict wave fields in such circumstances the most used are the method of coupled modes [1,2,3] and the parabolic equation technique in its various realizations [4−7].

In this paper we treat the problem of wave field calculations in the scalar and/or solid three-dimensional acoustic waveguide with properties slowly varying on horizontal coordinates and arbitrary on depth by the method of additive factorization. That enables to establish the equivalence of coupled modes two and one-way solutions or guided waves equations to the appropriate systems of abstract parabolic pseudodifferential equations and to get the iteration process to evaluate the transmitted and backscattered waves.

The algorithms of two- and one-way field computations based on high-order Pade approximations are proposed featuring the low computation cost at any given accuracy.

In Sec.I we introduce the operator form of Green's function of point harmonic source in layered waveguide, then we apply the method of additive factorization to the harmonic field calculation in three-dimensional waveguide with translational or rotational symmetry and formulate the Pade approximations based numerical algorithms for two- and one-way solutions.

In Sec.II we describe the similar technique to calculate fields with arbitrary time dependence.

In Sec.III the fully three-dimensional problem is treated in one-way on x and two-way on y manner.

In Sec.IV the two- and three-dimensional problems involving solid bottom are considered.

1 BASIS INVARIANT FORM OF GUIDED WAVES EQUATIONS AND ADDITIVE FACTORIZATION

Next exposition uses Riesz's definition [8] of function f of linear operator $\mathbf{T}$ ($\mathbf{I}$ - identity operator):

$$f(\mathbf{T}) = (2\pi i) \int_{\Gamma} f(\lambda)(\mathbf{T} - \lambda \mathbf{I})^{-1} d\lambda \tag{1}$$

where the integration path Γ encircles the spectrum of operator $\mathbf{T}$ in complex plane λ leaving him left.

Considering now the well known integral form of point harmonic source acoustical pressure field $p(x, y, z)$ in layered medium [9] one can see that it exactly conforms with

$$p(x, y, z) = i\pi H_0^{(1)}(\sqrt{\mathbf{T}(x^2 + y^2)})\delta(z - z_s), \tag{2}$$

where $\mathbf{T}$ is the transversal differential operator

$$\mathbf{T}q(z) = \rho D_z \rho^{-1} D_z q(z) + \omega^2 C^{-2} q(z) \tag{3}$$

with definition domain consisting of functions $q(z)$ smooth enough and satisfying the appropriate boundary conditions, for example

$$q(0) = 0, \ \max_{z>0} q(z)q^*(z) < const \tag{4}$$

Eq. (2) establishes the analogy between the two-dimensional Helmholtz equation

$$D_x^2 p + D_y^2 p + K^2 p = \delta(x, y) \tag{5}$$

with solution $H_0^{(1)}(\sqrt{K^2(x^2 + y^2)})$ and its three-dimensional case

$$D_x^2 p + D_y^2 p + \mathbf{T}p = \delta(x, y, z). \tag{6}$$

giving us a motivation for further investigation.

It is convenient to describe the wave propagation in heterogeneous media by the system of differential equations with coefficients depending only on the material properties - not on their derivatives - the system of linearized hydrodynamics equations. For the acoustical wave propagation in a three-dimensional liquid waveguide translationally invariable on y cartesian coordinate without currents it takes the form:

$$\begin{pmatrix} D_x & 0 & -i\omega\rho \\ D_z & -i\omega\rho & 0 \\ -i\omega\rho^{-1}C^{-2} & D_z & D_x \end{pmatrix} \begin{pmatrix} p \\ v_z \\ v_x \end{pmatrix} = \begin{pmatrix} f_x \\ f_z \\ V \end{pmatrix}$$

$$Y_0 p(x, 0) + v_z(x, 0) = 0, Y_H p(x, H) + v_z(x, H) = 0$$

$$(x, z) \in (-\infty, \infty) \times (0, H), \tag{7}$$

p being acoustical pressure, v_x, v_z-acoustical velocity cartesian components, f_x, f_z - cartesian components of external forces density, V-external volume velocity density, ρ-medium density, C-medium sound speed, all of them being functions of cartesian horizontal coordinate x or range and vertical coordinate z or depth. D_x and D_z are partial derivatives on x and z, ω - cyclic frequency, $i^2 = -1, Y_0$ and Y_H - acoustical admittances at upper and lower boundaries of waveguide, H - the maximum depth of the waveguide, taken into account. First two equations of (7) are the Newton equations, the third is the continuity equation. Deleting from (7) the vertical component of the acoustical velocity v_z, we obtain the operator form of (7):

$$\begin{pmatrix} D_x \mathbf{p} \\ D_x \mathbf{v} \end{pmatrix} = \begin{pmatrix} 0 & i\omega\mathbf{R} \\ i\omega\mathbf{B} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ \mathbf{v} \end{pmatrix} + \begin{pmatrix} \mathbf{f}_x \\ \mathbf{W} \end{pmatrix}$$

$$\mathbf{p}(x) = p(x, z), \mathbf{v}_x(x) = v_x(x, z),$$

$$\mathbf{R}\mathbf{v} = \rho(x, z) v_x(x, z),$$

$$\mathbf{B}\mathbf{p}(x) = (\rho^{-1}(x, z)C^{-2}(x, z) - D_z((i\omega)^2\rho(x, z))^{-1}D_z)p(x, z),$$

$$\mathbf{f}_z(x) = f_z(x, z),\ \mathbf{W}(x) = V(x, z) - D_z(i\omega\rho(x, z))^{-1} f_z(x, z)$$

$$Y_0 p(x, 0) + (i\omega\rho(x, 0))^{-1} D_z p(x, 0) = 0,$$

$$Y_H p(x, H) + (i\omega\rho(x, H))^{-1} D_z p(x, H) = 0 \tag{8}$$

Operator equations (8) are basis invariant form of guided waves equations [1,2]: assuming for the sake of simplicity $\mathbf{R} = \mathbf{I}$ and taking into account that the operator $\mathbf{B}$ in the local normal modes $\phi_l(x,z)$ basis has the diagonal form

$$\mathbf{B}(x) = \hat{\Phi}(x)\mathbf{C}^{-2}(x)\hat{\Phi}^{-1}(x), \hat{\Phi}(x) = row_l\{\vec{\phi}_l(x)\}, \mathbf{C}(x) = diag_l\{C_l\} \qquad (9)$$

C_l being the phase velocity of l-th local normal mode, we obtain for the local normal modes amplitudes $\mathbf{a}(x) = \hat{\Phi}^{-1}(x)\mathbf{p}(x)$, $\mathbf{b}(x) = \hat{\Phi}^{-1}(x)\mathbf{v}(x)$ the equations:

$$D_x a_l(x) - i\omega b_l(x) = -\sum_k \gamma_{lk}(x) a_k(x),,$$

$$D_x b_l(x) - i\omega C_l^{-2}(x) a_l(x) = -\sum_k \gamma_{lk}(x) b_l(x), \qquad (10)$$

where $\gamma_{lk} = \{\hat{\Phi}^{-1}(x)(D_x\hat{\Phi}(x))\}_{lk}$ are the local normal modes coupling coefficients.

We shall now reformulate eqs. (8) in the manner of Wentzel - Kramer - Brillouin method. Let us assume, that outside of interval $x \in (a,b)$ our waveguide is layered and all sources are situated inside this interval. Introducing the new local amplitudes of waves of two opposite directions:

$$\begin{pmatrix} \mathbf{p} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} \mathbf{Y}^{-1}(x) & \mathbf{Y}^{-1}(x) \\ \mathbf{Z}(x) & -\mathbf{Z}(x) \end{pmatrix} \begin{pmatrix} \mathbf{u}^+ \\ \mathbf{u}^- \end{pmatrix}$$

$$\mathbf{S}(x) = \left(\sqrt{\mathbf{R}(x)}\mathbf{B}(x)\sqrt{\mathbf{R}(x)}\right)^{1/2}$$

$$\mathbf{Y}(x) = \sqrt{\mathbf{S}(x)}\left(\sqrt{\mathbf{R}(x)}\right)^{-1}, \ \mathbf{Z}(x) = \left(\sqrt{\mathbf{R}(x)}\right)^{-1}\sqrt{\mathbf{S}(x)} \qquad (11)$$

and taking into account that outside of the interval (a,b) only outgoing waves must exist, we obtain by this process of additive factorization[10] the following system of equations:

$$\begin{pmatrix} \mathbf{P} & \mathbf{U} \\ \mathbf{L} & \mathbf{Q} \end{pmatrix} \begin{pmatrix} \mathbf{u}^+ \\ \mathbf{u}^- \end{pmatrix} = \begin{pmatrix} \mathbf{F}^+ \\ \mathbf{F}^- \end{pmatrix}$$

$$\mathbf{P} = D_x - i\omega\mathbf{S}(x) + 1/2\left(\mathbf{Z}^{-1}(x)(D_x\mathbf{Z}(x)) - \mathbf{Y}^{-1}(x)(D_x\mathbf{Y}(x))\right), \ \mathbf{u}^+(a) = 0$$

$$\mathbf{Q} = D_x + i\omega\mathbf{S}(x) + 1/2\left(\mathbf{Z}^{-1}(x)(D_x\mathbf{Z}(x)) - \mathbf{Y}^{-1}(x)(D_x\mathbf{Y}(x))\right), \ \mathbf{u}^-(b) = 0$$

$$\mathbf{L} = \mathbf{U} = -1/2\left(\mathbf{Z}^{-1}(x)(D_x\mathbf{Z}(x)) + \mathbf{Y}^{-1}(x)(D_x\mathbf{Y}(x))\right)$$

$$\begin{pmatrix} \mathbf{F}^+ \\ \mathbf{F}^- \end{pmatrix} = 1/2 \begin{pmatrix} \mathbf{Y} & \mathbf{Z}^{-1} \\ \mathbf{Y} & -\mathbf{Z}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{f}_x \\ \mathbf{W} \end{pmatrix}$$

$$(12)$$

wich can be solved by the Gauss-Zeidel iterations:

$$\mathbf{u}_k^+ = \mathbf{P}^{-1}(-\mathbf{U}\mathbf{u}_{k-1}^- + \mathbf{F}^+), \ \mathbf{u}_k^- = \mathbf{Q}^{-1}(-\mathbf{L}\mathbf{u}_k^+ + \mathbf{F}^-) \tag{13}$$

converging, if $\left|\mathbf{P}^{-1}\mathbf{U}\mathbf{Q}^{-1}\mathbf{L}\right| < 1$ for some norm $|*|$. If ρ and β are slowly varying with x, then $\mathbf{U}$ and $\mathbf{L}$ are small and for well defined Cauchy's operators $\mathbf{P}$ and $\mathbf{Q}$ the convergence of (13) is very probable. From eq.(13) we can see that the operators $\mathbf{L}$ and $\mathbf{U}$ describe the backscattering, $\mathbf{P}$ and $\mathbf{Q}$ - propagation (with terms $\pm i\omega\mathbf{S}$) and transmission through the inhomogeneities. Assuming now zero field as zero order approximation we obtain for the first order approximation of one-way positive direction wave $\mathbf{u}_1^+$:

$$\mathbf{P}\mathbf{u}_1^+ = \mathbf{F}^+ \tag{14}$$

yielding in the local normal modes basis the one-way guided waves equations [2]:

$$D_x c_l - i\omega C_l^{-1} c_l = -\sum_k \gamma_{lk}(C_l + C_k)(2C_lC_k)^{-1} + \{\hat{\Phi}^{-1}\mathbf{F}^+\}_l \tag{15}$$

Supposing the coupling coefficients γ_{lk} to be fast diminishing with $|l-k|$ and therefore letting

$$(C_l + C_k)/(2C_lC_k)^{-1} \approx 1 \tag{16}$$

neglecting so the difference of transmission from identity, we can obtain then in operator form the abstract parabolic equation

$$D_x\mathbf{u}_1^+ - i\omega\mathbf{S}(x)\mathbf{u}_1^+ = \mathbf{F}^+ \tag{17}$$

giving rise to the family of known PEs, according to the technique employed to approximate $\mathbf{S}$ and to solve the resulting approximated equation.

The aim of additive factorization (11,12) is to block-diagonalize the matrix of eq.(8) in layered case. If $\mathbf{B}$ is selfadjoint then this factorization conserves the energy flow form [11]:

$$\mathbf{p}^*\mathbf{v} = (\,\mathbf{Y}^{-1}(\mathbf{u}^+ \quad + \quad \mathbf{u}^-)\,)^*\,(\,\mathbf{Z}(\mathbf{u}^+ \quad - \quad \mathbf{u}^-)\,) = (\mathbf{u}^+)^*\mathbf{u}^+ - (\mathbf{u}^-)^*\mathbf{u}^- \tag{18}$$

and has a small enough nondiagonal $\mathbf{L}$ and $\mathbf{U}$, resulting in small backscattering and near to identity transmission. Moreover, this factorization provides the exact reciprocity of right and left going waves of first order - solutions of

$$\mathbf{P}\mathbf{u}_1^+ = \mathbf{F}^+ \tag{19}$$

and

$$\mathbf{Q}\mathbf{u}_1^- = \mathbf{F}^-, \tag{20}$$

The fundamental discussion of this question and Riccati's equation based formulation of (11,12) can be found in [11–14]. For underwater sound pressure calculations the factorization

$$\begin{pmatrix} \mathbf{p} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{Z}(x) & -\mathbf{Z}(x) \end{pmatrix} \begin{pmatrix} \mathbf{u}^+ \\ \mathbf{u}^- \end{pmatrix}$$

$$\mathbf{S}(x) = \begin{pmatrix} \mathbf{R}(x) & \mathbf{B}(x) \end{pmatrix}^{1/2}$$

$$\mathbf{Z}(x) = \mathbf{R}^{-1}(x)\mathbf{S}(x) = \mathbf{B}(x)\mathbf{S}^{-1}(x) \tag{21}$$

appears to be of some interest due to the simplicity of equality

$$\mathbf{p} = \mathbf{u}^{+} + \mathbf{u}^{-} \tag{22}$$

The equations, corresponding to (12) are now

$$\begin{pmatrix} \mathbf{P} & \mathbf{U} \\ \mathbf{L} & \mathbf{Q} \end{pmatrix} \begin{pmatrix} \mathbf{u}^{+} \\ \mathbf{u}^{-} \end{pmatrix} = \begin{pmatrix} \mathbf{F}^{+} \\ \mathbf{F}^{-} \end{pmatrix}$$

$$\mathbf{P} = D_x - i\omega\mathbf{S}(x) + 1/2\mathbf{Z}^{-1}(x)(D_x\mathbf{Z}(x)),\ \mathbf{u}^{+}(a) = 0$$

$$\mathbf{Q} = D_x + i\omega\mathbf{S}(x) + 1/2\mathbf{Z}^{-1}(x)(D_x\mathbf{Z}(x)),\ \mathbf{u}^{-}(b) = 0$$

$$\mathbf{L} = \mathbf{U} = -1/2\mathbf{Z}^{-1}(x)(D_x\mathbf{Z}(x)) \tag{23}$$

$$\begin{pmatrix} \mathbf{F}^{+} \\ \mathbf{F}^{-} \end{pmatrix} = 1/2 \begin{pmatrix} \mathbf{I} & \mathbf{Z}^{-1} \\ \mathbf{I} & -\mathbf{Z}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{f}_x \\ \mathbf{W} \end{pmatrix}$$

and lead under the same assumptions to the eq. (17).

To solve (17) on the x grid with step h it is convenient to use the algorithm

$$\mathbf{u}^{+}(x+h) = \exp(i\omega h\mathbf{S}(x))\mathbf{u}^{+}(x) \tag{24}$$

of exponential fitting type [4,15,16]. To approximate the propagation factor operator

$$\mathbf{E} = \exp(i\omega h\mathbf{S}(x)) \tag{25}$$

with any given accuracy we use the rational approximation: if

$$f(\lambda) \approx F_n(\lambda)/G_m(\lambda) \tag{26}$$

(F_n, G_m-polynomials of degrees n, m) in the vicinity of the operator's $\mathbf{S}$ spectrum then (1) gives

$$f(\mathbf{S}) \approx F_n(\mathbf{S})(G_m(\mathbf{S}))^{-1} \tag{27}$$

We construct the appropriate approximation to $\exp(ih\sqrt{\lambda})$ with the known Pade approximations [17]:

$$\exp(\lambda) = \left(M_n(\lambda^2) + \lambda N_n(\lambda^2) \right) \left(M_n(\lambda^2) - \lambda N_n(\lambda^2) \right)^{-1} + E_n(\lambda),$$

$$\sqrt{\lambda} = F_m(\lambda)(G_m(\lambda))^{-1} + H_m(\lambda) \tag{28}$$

M_n, N_n, F_m, G_m being polynomials with real coefficients, computed by recurrence, E_m, H_n- approximation errors known to fade with increasing m and n

in the complex plane without the negative half of real axes. Combining this approximations we get

$$\exp(ih\sqrt{\lambda}) = \frac{G_m(\lambda)M_n(-h^2\lambda) + ihF_m(\lambda)N_n(-h^2\lambda)}{G_m(\lambda)M_n(-h^2\lambda) - ihF_m(\lambda)N_n(-h^2\lambda)} =$$

$$= \prod_{k=1}^{k=n+m} (\lambda - \mu_k^*(h))(\lambda - \mu_k(h))^{-1} \tag{29}$$

μ_k being the roots of denominator in the above fraction, μ_k^* - their complex conjugated. We have computed μ_k for some m, n,h. All of them are situated in the IV quadrant of complex plane far enough from the spectrum of **S**, lying in the I and II quadrants, the fact leading to stability of action by **E**. The error of (29) is decreasing with n, m increasing, in the whole complex plane without the negative half of real axes,but practically important is to approximate the region of spectrum, corresponding to propagating local normal modes, lying in the vicinity of (1,0) in the upper halfplane. For example, $m = 4, n = 4, h = 4\pi$ give for local normal modes with Brillouin angles up to 60^0 the phase error less than $5 * 10^{-6}$rad enabling underwater sound propagation calculations to the range of $9000kM$ with frequencies up to $100Hz$ with absolute phase error less then $\pi/2$.

The computer implementation of the above technique needs a discretization of vertical coordinate z and an appropriate discrete approximation of **S**. This can be done by finite-differences techniques, Galerkin's method, Marchuk's equalities method and etc[10]. The common feature of such techniques is that the operator **B** can be approximated by the product $\hat{B}^{-1}\hat{A}$ of band matrices $\hat{B}$ and $\hat{A}$, while **R** is approximated by the diagonal matrix $\hat{R}$, giving for **E** with **S** from eq. (21):

$$\hat{E} \approx \prod_k (\hat{R}\hat{B}^{-1}\hat{A} - \mu_k^*)(\hat{R}\hat{B}^{-1}\hat{A} - \mu_k)^{-1} =$$

$$= \prod_k (\hat{I} - 2Im(\mu_k)(\hat{A} - \mu_k\hat{B}\hat{R}^{-1})^{-1}\hat{B}\hat{R}^{-1}) \tag{30}$$

- an easy to implement algorithm, including multiplication by band matrices and solving systems of equations with such matrices. Taking now into account, that the rational approximation to **E** is fully determined by the set of μ_k, we can include into our consideration also the case of admittances Y_0, Y_H depending on local normal mode phase velocity, and henceforth, on spectral parameter μ_k, as for underlying semispace with known properties. This assumption leads to the dependence of $\hat{A}$ and $\hat{B}$ on μ_k due to their dependence from boundary conditions in **B** while the above form of rational approximation to **E** remains unchanged.This feature is unique to the propagation factor algorithm (29,30).The another advantage of (29,30) is the aggregate approximation of eq. (17) and his solution resulting in lower calculations cost comparably to other known techniques.

The three-dimensional cylindrically-symmetric problem with the source located on the symmetry axis

$$\begin{pmatrix} D_r & 0 & -i\omega\rho \\ D_z & -i\omega\rho & 0 \\ -i\omega\rho^{-1}C^{-2} & D_z & r^{-1}D_r r \end{pmatrix} \begin{pmatrix} p \\ v_z \\ v_r \end{pmatrix} = \begin{pmatrix} 0 \\ f_z 2r^{-1}\delta(r) \\ V2r^{-1}\delta(r) \end{pmatrix}$$

$$Y_0 p(r,0) + v_z(r,0) = 0, Y_H p(r,H) + v_z(r,H) = 0,$$
$$(r,z) \in (0,\infty) \times (0,H), \tag{31}$$

may be treated in the same way: substituting $r = \exp(x)$,
$\mathbf{w} = \exp(x)v_r$ and deleting v_z, we get for $r > 0$

$$\begin{pmatrix} D_x \mathbf{p} \\ D_x \mathbf{w} \end{pmatrix} = \begin{pmatrix} 0 & i\omega\mathbf{R} \\ i\omega\exp(2x)\mathbf{B} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ \mathbf{w} \end{pmatrix} \tag{32}$$

and the one-way equation after returning to r variable

$$D_r \mathbf{p}_1^+ - i\omega\mathbf{S}(r)\mathbf{p}_1^+ + r^{-1}/2 = 0 \tag{33}$$

or

$$\mathbf{p}_1^+ = r^{-1/2}\mathbf{q}$$
$$D_r \mathbf{q}^+ - i\omega\mathbf{S}(r)\mathbf{q}^+ = 0 \tag{34}$$

Assuming the waveguide to be layered in such vicinity of the source, that the simplest asymptotic expansion $(2/i\pi\omega sr)^{1/2}\exp(i\omega sr)$ of Hankel function $H_0^{(1)}(\omega sr)$ holds on its outer border and taking into account,that this expansion satisfies (33), we can get the initial condition for $\mathbf{q}$ as [18]:

$$\mathbf{q}(0) = (\omega\mathbf{S}/2\pi i)^{-1/2} W(z) \tag{35}$$

We have developed the package of FORTRAN-coded routines to calculate the long-range underwater sound propagation in range-depth dependent environment accordingly (30,34,35). The performance of this package may be approximately evaluated as 50 floating points operations per one wavelength on x per one node in grid on z having the typical value of one fourth of the wavelength by sufficient accuracy for Brillouin angles up to 35^0. In layered waveguides the-computations results are practically identical to normal modes derived results, some verification in range-dependent environment may be found in [19].

The recent discussion [6,7] showed the usefulness of two-way algorithm implementing and including the transmission operators in one-way solution. The convenient algorithmic implementation of two-way problem can be got through the observation that we may from beginning use the medium model consisting of short layered waveguides intervals to get the additive factorization of eq. (8) or (31). Introducing the grid on $xa = x_0, \ldots x_l, \ldots, x_L = b$ and denoting

$$E_l = \exp(i\omega S_l(x_l - x_{l-1})),\ S_l = \sqrt{\mathbf{R}(x)\mathbf{B}(x)},\ Z_l = \mathbf{R}^{-1}(x_l)S_l \qquad (36)$$

we get, omitting the intermediate calculations, the operators $\mathbf{P}, \mathbf{Q}$ as two-diagonal matrices, $\mathbf{L}, \mathbf{U}$ as diagonal matrices:

$$\mathbf{P} = \begin{pmatrix} I & 0 & & & & \\ -I & \frac{1+Z_0^{-1}Z_1}{2} & & & & \\ 0 & -E_1 & \frac{1+Z_1^{-1}Z_2}{2} & & & \\ & & \cdots & & & \\ & & -E_{L-1} & \frac{1+Z_{L-1}^{-1}Z_L}{2} & 0 \\ & & & -E_L & \frac{1+Z_L^{-1}Z_{L+1}}{2} \end{pmatrix}$$

$$\mathbf{Q} = \begin{pmatrix} -\frac{1+Z_1^{-1}Z_0}{2} & E_1 & & \\ 0 & -\frac{1+Z_2^{-1}Z_1}{2} & E_2 & \\ & & \cdot & \\ & & -\frac{1+Z_{L+1}^{-1}Z_L}{2} & I \\ & & 0 & I \end{pmatrix}$$

$$\mathbf{L} = \begin{pmatrix} \frac{Z_1^{-1}Z_0-1}{2} & 0 & & \\ 0 & \frac{Z_2^{-1}Z_1-1}{2} & & \\ & \cdot & & \\ & & \frac{Z_{L+1}^{-1}Z_L-1}{2} & 0 \\ & & 0 & 0 \end{pmatrix}$$

$$\mathbf{U} = \begin{pmatrix} 0 & 0 & & \\ 0 & -\frac{Z_0^{-1}Z_1-1}{2} & & \\ & & \cdot & \\ & & -\frac{Z_{L-1}^{-1}Z_L-1}{2} & 0 \\ & & 0 & -\frac{Z_L^{-1}Z_{L+1}-1}{2} \end{pmatrix} \qquad (37)$$

enabling the straightforward implementation of Gauss-Zeidel iterations without computation of D_xZ. The action by Z and Z^{-1} can be fulfilled using Pade approximations. The one-way algorithm, including the transmission operator would take the form

$$\mathbf{u}_{l+1} = 2(1 + Z_l^{-1}Z_{l+1})^{-1}E_l\mathbf{u}_l \qquad (38)$$

yielding by additional approximation $D_z \approx 0$ in $Z_l$and Z_{l+1} the energy-conserving or impedance-matched PE of[6].

Consider now the sound field in a layered waveguide in presence of horizontal current with current speed $(U, 0, 0)$. The system of linearized hydrodinamics equations is now

$$\begin{pmatrix} D_x & -\rho U' & \rho(-i\omega + UD_x) \\ D_z & \rho(-i\omega + UD_x) & 0 \\ \rho^{-1}C^{-2}(-i\omega + UD_x) & D_z & D_x \end{pmatrix} \begin{pmatrix} p \\ v_z \\ v_x \end{pmatrix} = 0$$

$$p(x, 0) = 0, p(x, H) = 0,$$

where $U' = \partial U/\partial z$ and we neglected the term $\partial P/\partial z$. Solving this system for D_x derivatives, one gets

$$D_x \begin{pmatrix} p \\ v_z \\ v_x \end{pmatrix} = \begin{pmatrix} -i\omega C^{-1}MQ & \rho Q(-U' + UD_z) & i\omega\rho Q \\ -\rho^{-1}U^{-1}D_z & i\omega U^{-1} & 0 \\ i\omega\rho^{-1}C^{-2}Q & Q(-C^{-2}UU' + D_z) & i\omega C^{-1}MQ \end{pmatrix} \begin{pmatrix} p \\ v_z \\ v_x \end{pmatrix}$$

where $M = U/C$ is the Mach number and we denoted $Q = (1 - M^2)^{-1}$. To insulate the unacoustical mode of motion with $p = 0$ we introduce in place of v_x the new variable $w = (-i\omega)^{-1} \, (\, (-U' + UD_z)v_z + v_x \,)$ to get

$$D_x \begin{pmatrix} p \\ v_z \\ w \end{pmatrix} =$$

$$\begin{pmatrix} -i\omega C^{-1}MQ & 0 & i\omega\rho Q \\ -\rho^{-1}D_z & i\omega & 0 \\ i\omega\rho^{-1}C^{-2}Q - (-2U'/U + D_z)(i\omega\rho)^{-1}D_z & -2U'/U & i\omega C^{-1}MQ \end{pmatrix} \begin{pmatrix} p \\ v_z \\ w \end{pmatrix}$$

seeking the solution of this sytem of linear differential equations as

$$(p, v_z, w)^t = \exp(\Lambda x)(p, v_z, w)^t$$

we get for Λ matrix the equation

$$det \begin{pmatrix} i\omega\mathbf{U} - \Lambda & 0 & i\omega\mathbf{R} \\ -\rho^{-1}D_z & i\omega - U\Lambda & 0 \\ i\omega\rho^{-1}C^{-2}Q - (-2U'/U + D_z)(i\omega\rho)^{-1}D_z & -2U'/U & i\omega\mathbf{U} - \Lambda \end{pmatrix} = 0$$

or, subtracting the second row multiplied by $-2U'/U$ from the third

$$i\omega\mathbf{U} + \Lambda = -\omega^2\mathbf{R} \, (\, \mathbf{B} - 2U'/U \, (\, (i\omega - U\Lambda)^{-1} - (i\omega)^{-1} \,) \, (i\omega\rho)^{-1}D_z \,)$$

where we denoted

$$\mathbf{U} = C^{-1}M(1 - M^2)^{-1}$$

$$\mathbf{R} = \rho(1-M^2)^{-1}$$

$$\mathbf{B} = \rho^{-1}C^{-2}(1-M^2)^{-1} - D_z(i\omega\rho)^{-1}D_z$$

Supposing now U to be small and neglecting the commutators we get as the first order approximation for acoustical waves, propagating in both directions

$$\Lambda = -i\omega\mathbf{U} + QU'(i\omega)^{-1}D_z \pm i\omega\sqrt{\mathbf{RB}}$$

The third value of Λ is singular ($\sim U^{-1}$) and describes nonacoustical mode of medium motion. From this we get the equations of waves propagating from left to right (+) and from right to left (-)

$$D_x u_+ = i\omega\mathbf{U} + QU'(i\omega)^{-1}D_z + i\omega\sqrt{\mathbf{RB}})u_+$$

$$D_x u_- = i\omega\mathbf{U} + QU'(i\omega)^{-1}D_z - i\omega\sqrt{\mathbf{RB}})u_+$$

with the sound pressure

$$p = u_+ + u_-$$

In a uniform unbounded medium from above equations follows the well known dispersion equation

$$\omega = Ck + Uk$$

(k - wave number). Numerical solving of above equations may be fullfilled with the split-step algorithm analogous to the well known Hardin-Tappert technique:

$$u_+(x+h) = \exp(h(\mathbf{V} + i\omega\sqrt{\mathbf{RB}}))u_+(x) =$$

$$= \exp(h\mathbf{V}/2)\exp(ih\omega\sqrt{\mathbf{RB}})\exp(h\mathbf{V}/2)u_+(x) + O(h^3)$$

with $\mathbf{V}= i\omega\mathbf{U}+QU'(i\omega)^{-1}D_z$ and h being the step on the horizontal coordinate. In slowly range depending environment these equations are valid under the same presumptions as above.

2 ARBITRARY TIME DEPENDENCE

We shall start this section with the remark, that the Cauchy's operator spectrum cannot be situated in any final part of the complex plane [20], the fact establishing relationship between definition (1) and the well known definition of function of time derivative operator D_t, based on the Laplace transform $\mathbf{L}$ [21] :

$$f(D_t)\phi(t) = (2\pi i)\int_\Gamma f(\lambda)(D_t-\lambda)^{-1}\phi(t)d\lambda =$$

$$= \int_\Gamma d\lambda f(\lambda)\int_0^\infty d\tau\phi(\tau)\exp(\lambda(t-\tau)) =$$

$$= \int_\Gamma d\lambda f(\lambda) \exp(\lambda t) \int_0^\infty d\tau \phi(\tau) \exp(-\lambda \tau) =$$

$$= \mathbf{L}^{-1} \left[f(p) \quad \mathbf{L}[\phi] \quad (p) \right] (t) \tag{39}$$

where Γ must be any path encircling the infinity point of complex plane, for example, the Mellin's path $(c-i\infty, c+i\infty)$,if $f(p)\mathbf{L}[\phi](p)$ does not have irregular points on the right of this path.

Now we shall formulate the fundamental solutions of one and two-dimensional wave equation initial problems as outgoing on the space coordinate waves. For the problem on the real axes with zero initial conditions

$$D_x^2 u(x,t) - C^{-2} D_t^2 u(x,t) = -\delta(x)\delta(t) \tag{40}$$

the well known [22] fundamental solution

$$u(x,t) = C/2\theta(t - C^{-1}|x|) \tag{41}$$

can be written as if (40) would be a Helmholtz equation with wave number $iC^{-1}D_t$:

$$u(x,t) = -\left(2i(iC^{-1}D_t)\right)^{-1} \exp\left(i|x|(iC^{-1}D_t)\right)\delta(t) \tag{42}$$

Really, due to the property of Laplace transform

$$\exp(-\tau D_t)\phi(t) = \phi(t-\tau) \tag{43}$$

we get

$$-\exp\left(i|x|(iC^{-1}D_t)\right)\left((2i(iC^{-1}D_t)^{-1}\right)\delta(t) =$$

$$\exp\left(i|x|(iC^{-1}D_t)\right)C/2\theta(t) = C/2\theta(t - C^{-1}|x|) \tag{44}$$

The two-dimensional problem on the plane with zero initial conditions

$$D_x^2 u(x,y,t) + D_y^2 u(x,y,t) - C^{-2} D_t^2 u(x,y,t) = -\delta(x)\delta(y)\delta(t) \tag{45}$$

may be solved the same way:

$$u(x,y,t) = (2\pi)^{-1} K_0(C^{-1}\sqrt{x^2+y^2}D_t)\delta(t) \tag{46}$$

because of

$$u(x,y,t) = (2\pi)^{-1}\theta(t - C^{-1}\sqrt{x^2+y^2})/\sqrt{t^2 - C^{-2}(x^2+y^2)} =$$

$$= \mathbf{L}^{-1}\left[K_0(C^{-1}\sqrt{x^2+y^2}p)\right](t) \tag{47}$$

(see [23]).

Now, as in Sec.I, we apply the method of additive factorization on x coordinate to the the system of differential equations, governing acoustical wave propagation in a three-dimensional liquid waveguide translationally invariable on y cartesian coordinate:

$$\begin{pmatrix} D_x & 0 & \rho D_t \\ D_z & \rho D_t & 0 \\ \rho^{-1}C^{-2}D_t & D_z & D_x \end{pmatrix} \begin{pmatrix} p \\ v_z \\ v_x \end{pmatrix} = \begin{pmatrix} f_x \\ f_z \\ V \end{pmatrix}$$

$$Y_0 p(x,0,t) + v_z(x,0,t) = 0,\ Y_H p(x,H,t) + v_z(x,H,t) = 0,$$

$$p(x,z,0) = 0,\ v_z(x,z,0) = 0,\ v_x(x,z,0) = 0$$

$$(x,z) \in (-\infty,\infty) \times (0,H), \tag{48}$$

and get under the same assumptions the pseudodifferential equation, governing the one-way propagation

$$D_x \mathbf{p}^+ + D_t \sqrt{C^{-2} + \rho D_z \rho^{-1} D_t^{-2} D_z}\mathbf{p}^+ = F^+ \tag{49}$$

We can solve it the same way as one-way eq. (17)for harmonic field:

$$\mathbf{p}^+(x+h,t) = \mathbf{E}\mathbf{p}^+(x,t) \tag{50}$$

with the propagation factor operator $\mathbf{E}$ being now

$$\mathbf{E} = \exp(-hD_t\sqrt{C^{-2} + \rho D_[z]\rho^{-1}D_t^{-2}D_z}) =$$

$$= \int_{-i\infty}^{i\infty} dp \int_\Gamma \exp(-hp\sqrt{\lambda})(C^{-2} + \rho D_z \rho^{-1} p^{-2} D_z - \lambda)^{-1} d\lambda \tag{51}$$

the path Γ now encircling all spectra $\sigma(p)$ of transversal differential operator in the last integral. To apply the Pade approximations technique consider now the banded sources of type

$$V = Re(W(t)\exp(-i\omega_0 t)) \tag{52}$$

where the source envelope $W(t)$ varies slowly with respect to $\cos(\omega_0 t)$ so having the finite bandwidth $h\Delta\omega < \omega_0$. Assuming the solution to have the same type $Re(\mathbf{p}(t)\exp(-i\omega_0 t))$, we get for $\mathbf{E}$ with (48,51)

$\mathbf{E} = \exp(-h(D_t - i\omega_0)\sqrt{C^{-2} + \rho D_z \rho^{-1}(D_t - i\omega_0)^{-2} D_z})=$

$$= \int_0^{i\Delta\omega} dp \int_\Gamma \exp(-h(p-i\omega_0)\sqrt{\lambda})(C^{-2} + \rho D_z \rho^{-1}(p-i\omega_0)^{-2} D_z - \lambda)^{-1} d\lambda \tag{53}$$

Suppose now, that we have computed a set of $\mu_k(h,\omega_0,p)$ from (29) for $p \in (0, i\Delta\omega)$ and have approximated μ_k and $Im(\mu_k)$ by some rational fractions on

p $M_k(h,\omega_0,p)$ and $I_k(h,\omega_0,p)$. The discrete approximation to the transversal differential operator takes now the form

$$C^{-2}+\rho D_z\rho^{-1}(D_t-i\omega_0)^{-2}D_z=\hat{C}^{-1}(\hat{A}+\hat{B}(D_t-i\omega_0)^{-2}) \tag{54}$$

Taking into account the approximations (30), wich have now to be exact enough in the interval $(\omega_0,\omega_0+\Delta\omega)$, we get the rational approximation to $\mathbf{E}$ as

$$\hat{E}\approx\prod_k\left(\hat{I}-\frac{2I_k(h,\omega_0,D_t)}{(\hat{A}+\hat{B}(D_t-i\omega_0)^{-2}-M_k^*(h,\omega_0,D_t)\hat{C}\hat{R}^{-1}}\hat{C}R^{-1}\right) \tag{55}$$

- the production of infinite impulse response time-domain filters with matrix coefficients. This algorithm allows calculations marching not only the time, but also the spatial horizontal coordinate and therefore is much easier to computer implementing, than traditional time-marching algorithms. The discrete approximation of time derivative operator D can be got in various ways, among those we prefer the discretizations based on the Pade approximations of relationship

$$\tau D_t=\ln\Delta \tag{56}$$

easily derived from (43), τ being the time step, Δ - the time τ shift operator: $\Delta\phi(t)=\phi(t+\tau)$. For example, using the (0,1) Pade approximation of $\lambda^{-1}\ln(1+\lambda)$, we get the widely used in filters design [24] approximation

$$D_t=2\tau^{-1}(\Delta-1)(\Delta+1)^{-1} \tag{57}$$

The higher degrees of Pade approximation give us the more exact discretizations.

The system of equations (48) is valid for the fluids without viscosity or other dissipation processes. For example, if only the one relaxation process exists in the medium, the eqs. (48) take form [25]

$$\begin{pmatrix} D_x & 0 & \rho D_t \\ D_z & \rho D_t & 0 \\ \rho^{-1}(1+\tau_r D_t)(C_1^2+C_2^2\tau_r D_t)^{-1}D_t & D_z & D_x \end{pmatrix}\begin{pmatrix} p \\ v_z \\ v_x \end{pmatrix}=\begin{pmatrix} f_x \\ f_z \\ V \end{pmatrix} \tag{58}$$

τ_r being the relaxation characteristic time, C_1- the sound velocity for small frequencies, C_2- for high frequencies, but if the signal is banded, the complex values of C in (48) may supply a good approximation.

3 THE THREE-DIMENSIONAL HETEROGENEOUS LIQUID WAVEGUIDE

Consider now the system of differential equations, governing harmonic acoustical wave propagation in a three-dimensional liquid waveguide

$$\begin{pmatrix} D_x & 0 & 0 & -i\omega\rho \\ D_y & 0 & -i\omega\rho & 0 \\ D_z & -i\omega\rho & 0 & 0 \\ -i\omega\rho^{-1}C^{-2} & D_z & D_y & D_x \end{pmatrix} \begin{pmatrix} p \\ v_z \\ v_y \\ v_x \end{pmatrix} = \begin{pmatrix} f_x \\ f_z \\ f_y \\ V \end{pmatrix}$$

$$Y_0 p(x,y,0) + v_z(x,y,0) = 0, Y_H p(x,y,H) + v_z(x,y,H) = 0$$

$$(x,y,z) \in (-\infty,\infty) \times (-\infty,\infty) \times (0,H), \tag{59}$$

Deleting, as in Sec.I from (59) the vertical and shear components of the acoustical velocity v_z and v_y , we obtain the operator form of (59):

$$\begin{pmatrix} D_x \mathbf{p} \\ D_x \mathbf{v} \end{pmatrix} = \begin{pmatrix} 0 & i\omega\mathbf{R} \\ i\omega\mathbf{B} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ \mathbf{v} \end{pmatrix} + \begin{pmatrix} \mathbf{f}_x \\ \mathbf{W} \end{pmatrix}$$

$$\mathbf{B}\mathbf{p}(x) = (\rho^{-1}C^{-2} - D_y((i\omega)^2\rho)^{-1}D_y - D_z((i\omega)^2\rho)^{-1}D_z)p(x,y,z) \tag{60}$$

Assuming the slowness of medium properties dependence on x, we get with the factorization (21,23) the one-way equation

$$D_x \mathbf{p}_1^+ - i\omega\mathbf{S}(x)\mathbf{p}_1^+ = 0$$

$$\mathbf{S} = (\, C^{-2} \quad - \quad \rho D_y((i\omega)^2\rho)^{-1}D_y \quad - \quad \rho D_z((i\omega)^2\rho)^{-1}D_z\,)^{1/2} \tag{61}$$

and approximate the propagation factor as

$$\hat{E} \approx \prod_k \left(\hat{I} - \quad 2Im(\mu_k) \left(C^{-2} - \rho D_y((i\omega)^2\rho)^{-1}D_y - \rho D_z((i\omega)^2\rho)^{-1}D_z - \mu_k\right)^{-1} \right) \tag{62}$$

From this we see, that for each step on x we need to solve some two-dimensional Helmholtz-type boundary problems in highly absorbing medium (all $Im\mu_k < 0$, see Sec.I) of type

$$\begin{pmatrix} D_y & 0 & -i\omega\rho \\ D_z & -i\omega\rho & 0 \\ -i\omega\rho^{-1}(C^{-2} - \mu_k) & D_z & D_y \end{pmatrix} \begin{pmatrix} p \\ v_z \\ v_y \end{pmatrix} = \begin{pmatrix} f_x \\ f_z \\ V \end{pmatrix}$$

$$Y_0 p(y,0) + v_z(y,0) = 0, Y_H p(y,H) + v_z(y,H) = 0 \tag{63}$$

This can be done by applying the full Gauss-Zeidel iterative process (13) if the medium properties dependence on y is also slow.

4 THE SOLID BOTTOM CONSIDERATIONS

The harmonical acoustical wave propagation in an isotropic elastic medium is governed by the Newton's and Gook's lows

$$i\omega\rho v_\alpha + \partial\sigma_{\alpha\beta}/\partial x_\beta = 0, i\omega\sigma_{\alpha\beta} + \lambda\delta_{\alpha\beta}\partial v_\gamma/\partial x_\gamma + \mu(\partial v_\alpha/\partial x_\beta + \partial v_\beta/\partial x_\alpha) = 0 \tag{64}$$

Here σ are stresses, v - velocities, λ and μ - Lame coefficients,
all of them depending on all cartesian coordinates x, y and z,
α, β, γ denote x or y or z. Arranging the unknown quantities in the order of $(\sigma_{xx}, v_z, \sigma_{xz}, v_x, \sigma_{xy}, v_y, \sigma_{yy}, \sigma_{yz}, \sigma_{zz})^t$ and the equations in the order of Newton's low for $v_x(N : v_x)$, Gook's low for $\sigma_{xz}(G : \sigma_{xz})$, $N : v_z$, $G : \sigma_{xx}$, $N : v_y$, $G : \sigma_{xy}$, $G : \sigma_{yy}$, $G : \sigma_{yz}$, $G : \sigma_{zz}$ we wright them in the form

$$\begin{pmatrix} D_x & 0 & D_z & i\omega\rho & D_y & 0 & 0 & 0 & 0 \\ 0 & \mu D_x & i\omega & \mu D_z & 0 & 0 & 0 & 0 & 0 \\ 0 & i\omega\rho & D_x & 0 & 0 & 0 & 0 & D_y & D_z \\ i\omega & \lambda D_z & 0 & (\lambda+2\mu)D_x & 0 & \lambda D_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_x & i\omega\rho & D_y & D_z & 0 \\ 0 & 0 & 0 & \mu D_y & i\omega & \mu D_x & 0 & 0 & 0 \\ 0 & \lambda D_z & 0 & \lambda D_x & 0 & (\lambda+2\mu)D_y & i\omega & 0 & 0 \\ 0 & \mu D_y & 0 & 0 & 0 & \mu D_z & 0 & i\omega & 0 \\ 0 & (\lambda+2\mu)D_z & 0 & \lambda D_x & 0 & \lambda D_y & 0 & 0 & i\omega \end{pmatrix} \times$$

$$\times (\sigma_{xx}, v_z, \sigma_{xz}, v_x, \sigma_{xy}, v_y, \sigma_{yy}, \sigma_{yz}, \sigma_{zz})^t = 0 \tag{65}$$

Supposing now the medium to be a waveguide $(x, z) \in (-\infty, \infty) \times (0, H)$ with translational symmetry on y cartesian axis and deleting last three stresses by combining 3,4 and 9, 5 and 8 equations, we get

$$\begin{pmatrix} D_x & 0 & D_z & i\omega\rho & 0 & 0 \\ 0 & D_x & i\omega/\mu & D_z & 0 & 0 \\ D_z\frac{\lambda}{\lambda+2\mu} & i\omega\rho + D_z\frac{4\mu(\lambda+\mu)}{i\omega(\lambda+2\mu)} & D_z & D_x & 0 & 0 \\ \frac{i\omega}{\lambda+2\mu} & \frac{\lambda}{\lambda+2\mu}D_z & 0 & D_x & 0 & 0 \\ 0 & 0 & 0 & 0 & D_x & i\omega\rho - D_z\frac{\mu}{i\omega}D_z \\ 0 & 0 & 0 & 0 & \frac{-1}{\mu} & D_x \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ v_z \\ \sigma_{xz} \\ v_x \\ \sigma_{xy} \\ v_y \end{pmatrix} = 0 \tag{66}$$

The wave $(\sigma_{xy}, v_y)^t$ is seen to propagate independently. Introducing **R** and **B** operators as

$$\mathbf{R} = \begin{pmatrix} D_z & i\omega\rho \\ \frac{i\omega}{\mu} & D_z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}^{-1} \begin{pmatrix} D_z & i\omega\rho \\ i\omega & \mu D_z \end{pmatrix} = M^{-1}R$$

$$\mathbf{B} = \begin{pmatrix} D_z \frac{\lambda}{\lambda+2\mu} & i\omega\rho + D_z \frac{4\mu(\lambda+\mu)}{i\omega(\lambda+2\mu)} D_z \\ \frac{i\omega}{\lambda+2\mu} & \frac{\lambda}{\lambda+2\mu} D_z \end{pmatrix} \tag{67}$$

we can write the equations for shear $s = (\sigma_{xz}, v_x)^t$ and compression $c = (\sigma_{xx},\ v_z)^t$ waves in the form

$$\begin{pmatrix} D_x c \\ D_x s \end{pmatrix} = \begin{pmatrix} 0 & -\mathbf{R} \\ -\mathbf{B} & 0 \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} \tag{68}$$

All considerations of Sec. I are applicable with minor difference: the spectrum of $\mathbf{S}$ will now occupy the I,II,III quadrants of the complex plane, forcing the cut of the squareroot in (28) follow the $(0, -i\infty)$ ray. The appropriate squareroot's Pade approximation can be easily derived:

$$\sqrt{\lambda} = \exp(i\pi/4) F_n(\exp(-i\pi/2)\lambda)/G_n(\exp(-i\pi/2)\lambda) \tag{69}$$

Even if the layers of fluid with $\mu = 0$ are present in the model, the rational approximation (29,30) of propagation operator holds due to its nonsingular form

$$\hat{E} \approx \prod_k (\, M^{-1} R\mathbf{B} - \mu_k^* \,)\, (\, M^{-1} R\mathbf{B} - \mu_k \,)^{-1} =$$
$$= \prod_k (\, \hat{I} - \quad 2Im(\mu_k)(R\mathbf{B} - \mu_k M)^{-1} M \,) \tag{70}$$

enabling an uniform implementation of the algorithm.

5 CONCLUSION

We have used operator notation of fluid and solid acoustic's equations to get the system of pseudodifferential equations, governing the two- and one-way propagation of sound in range-depth and $3D$ dependent environment, outlined its physical sense and proposed effective numerical algorithms to solve it for two- and one-way propagating fields, including arbitrary time dependence and banded signals.

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