Title: A Galerkin technique for Sturm-Liouville eigenvalue problems with complex and discontinuous coefficients, arising in underwater acoustics, Richard B. Evans

Abstract: The depth separated wave equation in underwater acoustics is a non-self-adjoint Sturm-Liouville eigenvalue problem with complex and discontinuous coefficients. A Galerkin approach, based on a simple real problem with appropriate derivative discontinuities, leads to a complex symmetric generalized matrix eigenvalue problem. The eigenvector solutions are found to satisfy a weighted biorthogonality condition that is desirable for coupled mode applications in laterally varying ocean waveguides. Other advantages are also discussed.

Introduction:

The depth separated wave equation, for the harmonic acoustic pressure, has the form of a Sturm-Liouville problem[1] given by

$$\frac{d}{dz}\left[\frac{1}{\rho(z)}\frac{d\phi}{dz}\right]+\frac{k^2(z)}{\rho(z)}\phi=\frac{\lambda\phi}{\rho(z)} \tag{1}$$

where λ is a separation constant whose values are determined by the coefficients in the differential equation and the boundary conditions at the ends of the interval $0 \leq z \leq H$. The boundary conditions used are $\phi(0)=0$ and $\phi(H)=0$. The boundary conditions correspond to a free (pressure release) surface at $z=0$ and H. The function ρ is the density and $k=(\omega/c)[1+i(\alpha/40\pi\log_{10}e)]$ is the complex wave number, where ω is the circular frequency, c is the sound speed and α is the attenuation in dB per wavelength. The functions ρ and k^2 can be discontinuous at the bottom of the water column, at the interface $z=h$ where $h<H$. The continuity of pressure and particle velocity dictate that $\phi(h^-)=\phi(h^+)$ and

$$\frac{1}{\rho(h^-)}\frac{d\phi(h^-)}{dz}=\frac{1}{\rho(h^+)}\frac{d\phi(h^+)}{dz} \tag{2}$$

where the superscripts $-$ and $+$ indicate limits from above and below, respectively. The Sturm-Liouville eigenvalue problem consisting of Eq. (1) and associated boundary and interface conditions is non-self-adjoint because the wave number squared is complex. The eigenvalues and eigenfunctions are generally complex. It is assumed that the attenuation is small enough so that the eigenvalues are simple roots of the characteristic equation.

The Galerkin Method:

The challenge of finding the complex eigenvalues and eigenfunctions will be met by using the Galerkin method[2]. The Galerkin method replaces the differential eigenvalue problem in Eq. (1) with a matrix eigenvlaue problem. The matrix eigenvlaue problem is obtained by projecting Eq. (1) onto a finite dimensional subspace spanned by a set of known basis functions.

Suppose that the complex wave number squared and density, in Eq. (1), are defined in the water and bottom, separately, using

$$c(z)=\begin{cases} c_w(z), & 0\leq z\leq h \\ c_b(z), & h<z\leq H \end{cases}, \quad \alpha(z)=\begin{cases} \alpha_w(z), & 0\leq z\leq h \\ \alpha_b(z), & h<z\leq H \end{cases},$$

and

$$\rho(z) = \begin{cases} \rho_w(z), & 0 \le z \le h \\ \rho_b(z), & h < z \le H \end{cases} .$$

An example of a sound speed and density profile is shown, to the right of the depth axis, in Fig. 1. A simple problem, with no attenuation, will be constructed using a piecewise constant sound speed and density profile defined by

$$c_0(z) = \begin{cases} c_w, & 0 \le z \le h \\ c_b, & h < z \le H \end{cases} \quad and \quad \rho_0(z) = \begin{cases} \rho_w, & 0 \le z \le h \\ \rho_b, & h < z \le H \end{cases}$$

as shown on the left side of the depth axis, in Fig. 1.

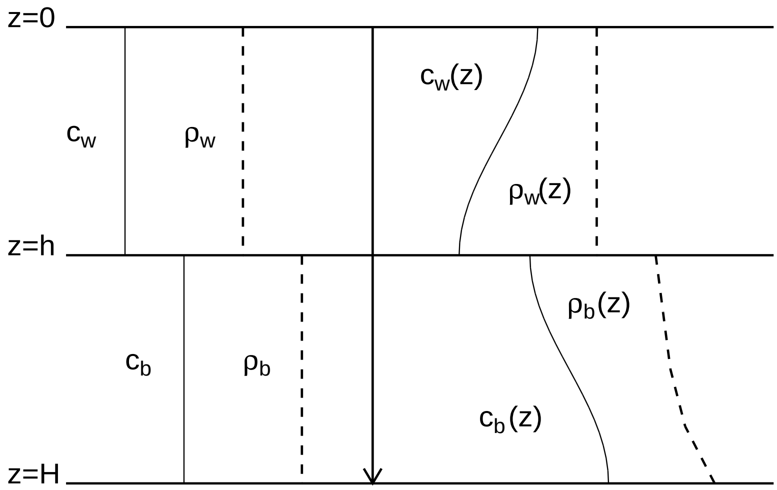

Figure 1. Sound speed and density profiles for the application of the Galerkin method.

The source of the basis functions, used in the Galerkin method, is the set of real eigenfunctions of the Strum-Liouville problem given by

$$\frac{d}{dz}\left[\frac{1}{\rho_0(z)}\frac{d\psi_m}{dz}\right] + \frac{k_0^2(z)}{\rho_0(z)}\psi_m = \frac{\gamma_m \psi_m}{\rho_0(z)}, \quad m = 1, \infty \tag{3}$$

where the coefficients are defined using the piecewise constant profiles already introduced and the wave number $k_0 = \omega/c_0$ is real. The boundary conditions and interface conditions associated with Eq. (3) are taken to be the same as the boundary conditions and interface conditions associated with Eq. (1) to facilitate the convergence of the Galerkin expansion. In particular, it is important that the real eigenfunctions have the same derivative discontinuity as specified in Eq. (2), to avoid Gibbs oscillations[3]. Consequently, we will take $\rho_w = \rho_w(h^-)$ and $\rho_b = \rho_b(h^+)$. The choices of c_w and c_b are not as important, but we suggest $c_w = avg\{c_w(z), 0 \le z \le h\}$ and $c_b = avg\{c_b(z), h \le z \le H\}$, for general applications. The same set of basis functions was used by Evans and

Gilbert[4] to solve Eq. (1) when only the sound speed profile varied continuously, in two layers.

The Sturm-Liouville eigenvalue problem consisting of Eq. (3) and the associated boundary and interface conditions is self-adjoint. The eigenvalues γ_m are simple roots of the characteristic equation that can be found by searching along the real axis. The real eigenfunctions ψ_m are orthogonal[5] and can be normalized such that

$$\int_0^H \frac{\psi_n(z)\psi_m(z)}{\rho_0(z)} dz = \delta_{n,m} \tag{4}$$

where $\delta_{n,n} = 1$ and $\delta_{n,m} = 0$ if $n \neq m$. Since the coefficients in Eq. (3) are piecewise constant, the eigenfunctions ψ_m are sine waves, in the water and bottom layers. The arguments of the sine waves are determined, in each layer, by the eigenvalues γ_m. The integrals, required to effect the normalization in Eq. (4), can be computed, analytically, in closed form. Equation (4) is the weighted orthogonality condition for the real eigenfunctions.

An eigenfunction of Eq. (1) is approximated by the truncated expansion $\phi = \sum_{m=1}^{M} u_m \psi_m$ where M is chosen to obtain an adequate range of modal angles, in the eigenfunction solutions of Eq. (1). A rule of thumb[6] is to make M twice as large as the total of the number of propagating and bottom interacting modes needed to represent the acoustic pressure. Substituting the truncated expansion into Eq. (1), requiring that the residual error be orthogonal to each of the $\psi_m, m = 1, M$, and applying integration by parts yields

$$\sum_{m=1}^{M} A_{n,m} u_m = \lambda \sum_{m=1}^{M} B_{n,m} u_m \tag{5}$$

where

$$A_{n,m} = \int_0^H \frac{k^2(z)\psi_n(z)\psi_m(z)}{\rho(z)} dz - \int_0^H \frac{1}{\rho(z)} \frac{d\psi_n(z)}{dz} \frac{d\psi_m(z)}{dz} dz$$

and

$$B_{n,m} = \int_0^H \frac{\psi_n(z)\psi_m(z)}{\rho(z)} dz.$$

When k^2 is linear and ρ is exponential or constant, in multiple layers, the above integrals can be computed in closed form. Note that the real eigenfunctions are not orthogonal when weighted by $1/\rho$. Other advantages are also discussed.

A Matrix Eigenvalue Problem:

Suppose A is the MxM matrix whose entries are $A_{n,m}$ and B is the MxM matrix whose entries are $B_{n,m}$. If the column vector $\vec{u}$ is defined by $\vec{u} = col(u_m, m = 1, M)$, then Eq. (5) can be written as

$$A\vec{u} = \lambda B\vec{u} \tag{6}$$

where A is a complex symmetric (not Hermitian) matrix and B is a real symmetric matrix. Equation (6) is called a generalized matrix eigenvalue problem because B is not, necessarily, the identity matrix. The eigenvalues of Eq. (6) approximate the eigenvalues of Eq. (1). The eigenvectors of Eq. (6) provide the expansion coefficients for approximates of the eigenfunctions of Eq. (1).

Equation (6) can be turned into a standard matrix eigenvalue problem by multiplying on the left by B^{-1}, but this destroys the symmetry of the problem. If B is positive definite, then Eq. (6) can be replaced by the complex symmetric eigenvalue problem

$$C\vec{w} = \lambda\vec{w} \tag{7}$$

where $C = L^{-1}A(L^T)^{-1}$, $\vec{w} = L^T\vec{u}$, and $B = LL^T$ is the Cholesky decomposition of B. The superscript T stands for transpose and L is a lower triangular matrix, making the calculation of C relatively easy[7]. Note that $\vec{u}^T$ is a row vector and its product with the column vector $\vec{v} = col(v_j, j = 1, M)$ is the scalar $\vec{u}^T\vec{v} = \sum_{j=1}^{M} u_j v_j$.

To see that B is positive definite, let $\vec{x} = col(x_j, j = 1, M)$ be a real vector and define the function $f = \sum_{j=1}^{M} x_j\psi_j$. It follows that

$$\int_0^H \frac{f^2(z)}{\rho(z)}dz = \vec{x}^T B\vec{x} \quad .$$

Consequently, $\vec{x}^T B\vec{x} > 0$ unless $\vec{x} = \vec{0}$.

We will assume that the complex symmetric matrix C in Eq. (7) is diagonalizable[8]. In this case, $C = W\Lambda W^T$ where $\Lambda = diag(\lambda_m, m = 1, M)$ and W is an orthogonal matrix (not unitary) such that $W^TW = I$ is the MxM identity matrix. The columns of $W = row(\vec{w}_m, m = 1, M)$ are the eigenvectors of C. The orthogonality of W implies that $\vec{w}_n^T\vec{w}_m = \delta_{n,m}$. The eigenvectors of Eq. (6) are recovered by solving $L^T\vec{u}_m = \vec{w}_m$ and consequently $\vec{u}_n^T B\vec{u}_m = \delta_{n,m}$.

The complex symmetric matrix eigenvalue problem in Eq. (7) can be tackled by several methods. The Jacobi technique, due to Anderson and Loizou[9], is recommended here. The Jacobi technique is not the most efficient, but it is simple and foolproof, assuming that C is diagonalizable. The foolproof feature is important in a generally applicable numerical code.

Biorthogonality:

When the stepwise coupled mode procedure[10] is applied to a laterally varying ocean waveguide it is necessary to expand the complex pressure $p = \sum_{m=1}^{M} p_m\phi_m$ in terms of the complex eigenfunctions of Eq. (1). The weighted biorthogonality property of the complex eigenfunctions, given by

$$\int_0^H \frac{\phi_n(z)\phi_m(z)}{\rho(z)}dz = \delta_{n,m} \quad , \tag{8}$$

is needed to justify the projection

$$p_m = \int\limits_0^H \frac{p(z)\phi_m(z)}{\rho(z)} dz$$

used in the stepwise coupled mode procedure. Although Eq. (8) is directly analogous to the orthogonality condition in Eq. (4), the term biorthogonality is used because of the non-self-adjoint nature[11] of Eq. (1). In short, it means that the complex conjugate that would normally appear on one of the complex functions in the integrand is omitted. This conforms to the definition of the scalar product of two complex vectors, already introduced. The biorthogonality of the Galerkin approximates of the complex eigenfunctions is demonstrated as follows.

Consider the two approximate eigenfunctions of Eq. (1) given by $\phi_m = \sum_{j=1}^{M} u_{m,j}\psi_j$ and $\phi_n = \sum_{j=1}^{M} u_{n,j}\psi_j$. The integral in Eq. (8) can be evaluated by substitution of these two expansions to obtain

$$\int\limits_0^H \frac{\phi_n(z)\phi_m(z)}{\rho(z)} dz = \vec{u}_n^T B \vec{u}_m \quad .$$

The biorthogonality of the approximate eigenfunctions follows because $\vec{u}_n^T B \vec{u}_m = \delta_{n,m}$.

Conclusions:

The Galerkin method provides a relatively simple procedure for solving the non-self-adjoint Sturm-Liouville problem in Eq. (1). The solutions are useful when the stepwise coupled mode procedure is applied to a range dependent acoustic waveguide. The utility comes from the fact that the coupling integrals, between complex eigenfuntions or modes in regions of different depths and environmental parameters, can be computed in closed form. The coupling integrals between real eigenfuntions in different regions can be computed analytically since the integrands are products of sine waves. Only three contiguous depth integrals are needed at a change in the bottom depth. The coupling integrals between the complex eigenfuntions are found by matrix multiplication, using the Galerkin expansions.

The Gibbs oscillations that were avoided in the computation of the complex eigenfunctions re-appear between adjacent regions, but these oscillations are a valid feature of the stepwise coupled mode solution that subside away from the corners of the steps.

The rule of thumb regarding the choice of M does not answer the fundamental question regarding the convergence of the Galerkin expansion as $M \to \infty$. The convergence of the Galerkin expansion and the implicit assumption that the eigenvalues of Eq. (1) are simple roots of the characteristic equation would likely assure that the complex symmetric matrix C in Eq. (7) is diagonalizable. These issues require further investigation.

References:

[1]F. B. Jensen, W. A. Kuperman, M. B. Porter and H. Schmidt, *Computational Ocean Acoustics*, p. 258 (AIP Press, New York, 1994).

[2]J. P. Boyd, *Chebyshev and Fourier Spectral Methods*, Second Edition (Revised), pp. 67-80 (Dover, New York, 2001).

[3]D. Gottlieb and S. A. Orszag, *Numerical Analysis of Spectral Methods*, CBMS-NSF Regional Conference Series in Applied Mathematics, pp. 29-32 (Society for Industrial and Applied Mathematics, Philadelphia, 1977).

[4]R. B. Evans and K. E. Gilbert, "Acoustic propagation in an refracting ocean waveguide with an irregular interface," Comp. Maths. Appl. 11, 795-805 (1985).

[5]J. D. Pryce, *Numerical Solutions of Sturm-Liouville Problems*, p. 23 (Oxford University Press, New York, 1993).

[6]J. P. Boyd, p. 132.

[7]W. H. Press, S. A. Teukolsky, W. T Vettering and B. P. Flannery, *Numerical Recipes in FORTRAN*, Second Edition, p. 455 (Cambridge University Press, New York, 1992).

[8]R. A. Horn and C. R. Johnson, *Matrix Analysis*, pp. 211-212 (Cambridge University Press, New York, 1985).

[9]P. J. Anderson and G. Loizou, "A Jacobi type method for complex symmetric matrices," Numer. Math. 25, 347-363 (1976).

[10]F. B. Jensen, et al., pp. 299-303.

[11]E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, p. 310 (McGraw-Hill, New York, 1955).